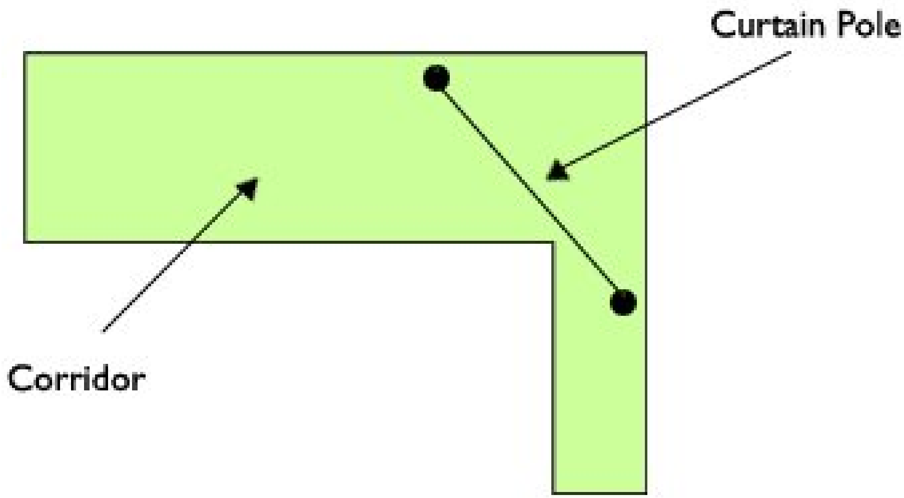
The Curtain pole problem

When rooms are decorated, it can be difficult to move furniture and fittings along the corridors into the room that is being decorated. In this problem, we are going to consider moving a curtain pole (or other rod) along a corridor which has a right angled bend in it. We want to determine whether it is possible to carry the pole round this corner.



Often modellers are presented with a problem that is imprecise or not well defined, so the first step in the modelling cycle is to clarify the purpose of the model, often by further discussion with the customer!

Specify the purpose

Here a more detailed question is:

What is the maximum length of pole that can be taken round a right-angled corner, if the two parts of corridor are different widths?

Next, try to get a feel for the problem by asking yourself questions and perhaps drawing a diagram. (This is always a good start for a geometrical problem so that you can see what is happening!)

Try this now. Assuming the pole is held horizontally, what are the physical restrictions on moving the pole round the corner? Can you draw a diagram which illustrates this limiting position?

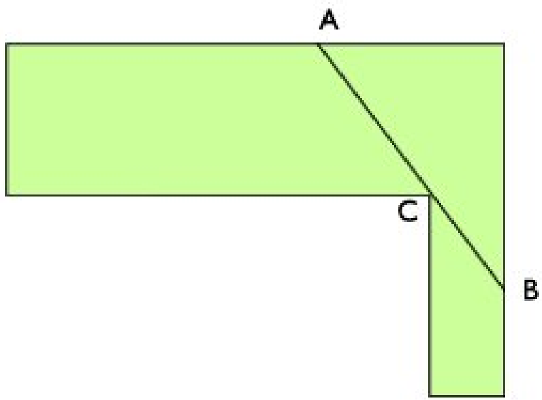


Figure 1: Illustrating the limiting position

The line AB represents the space available for the pole to be moved in. From this diagram, you can see that the length AB will depend on the angles the line makes with the walls and the widths of the two corridors. The corridor widths are fixed for a particular building, but the angle can vary.

By drawing lines on your diagram that touch the walls of the two corridors and the corner,

convince yourself that the length of this line varies as you change the angle.

We've made some progress here – we've looked at a simpler case of the original problem and have now clarified the mathematical question that we need to investigate:

What is the MINIMUM length of the line AB in Figure 1?

The next stage in the modelling cycle is to:

Create the model

Make assumptions first

The mathematical model will be based on these assumptions, so here are a few to start with:

- The rod is kept horizontal as it is moved round the bend.
- The rod is assumed to have negligible width as the width will be small compared to the length of the rod.
- The walls of the corridor in each section are perpendicular to the floor.

Notice how all these assumptions make the model simpler to solve. However, we may need to revise these assumptions later and make a second trip round the modelling cycle.

Can you think of any further assumptions?

Choose the variables

We now need to define some variables (with appropriate units and symbols) to describe the problem mathematically. One variable, L, could be the length of the line AB, measured in metres.

What other variables should be defined?

The next stage of the modelling cycle is:

Do the mathematics

Remember that the problem is to find the MINIMUM length of L as θ varies. What mathematical techniques have you met before, that help find minimum values?

We would need to find an expression for L in terms of θ, a, and b and then differentiate with respect to θ.

Find an expression for L. (Hint: Find angle DCA in terms of θ. Then find the lengths AC and CB and hence the length of AB.)

Now, determine the value of which minimises L. You can use the quotient or composite function rules (or standard results for sec and cosec) to differentiate L with respect to θ.

Then find the values of θ for which $\dfrac{dL}{d\theta}$ is zero.

But does this value of θ give a minimum value for L?

$$\text{We know that } \frac{dL}{d\theta} = \frac{-b\cos\theta}{\sin^2\theta} + \frac{a\sin\theta}{\cos^2\theta} = \frac{a\cos\theta}{\sin^2\theta}\left(\tan^3\theta - \frac{b}{a}\right).$$

If $0 < \theta < \frac{\pi}{2}$ then $\tan\theta > 0$.

So when $\tan\theta < \left(\frac{b}{a}\right)^{\frac{1}{3}}$, $\dfrac{dL}{d\theta} < 0$ and when $\tan\theta > \left(\frac{b}{a}\right)^{\frac{1}{3}}$, $\dfrac{dL}{d\theta} > 0$.

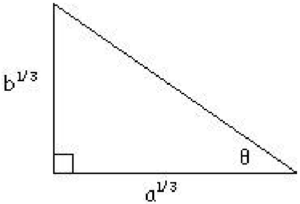
By the First Derivative Test, this shows that L has a minimum value when $\theta = \tan^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Alternatively, you can apply the Second Derivative Test to show that $\dfrac{d^2L}{d\theta^2} > 0$ when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Now we can express L in terms of a and b, the widths of the corridor sections.

As $0 < \theta < \frac{\pi}{2}$, we may use the properties of a right-angled triangle.

$$\begin{aligned}
\text{Length of hypotenuse} &= \left[(a^{\frac{1}{3}})^2 + (b^{\frac{1}{3}})^2\right]^{\frac{1}{2}} \\
&= \left[a^{\frac{2}{3}} + b^{\frac{2}{3}}\right]^{\frac{1}{2}}. \\
\text{So } \frac{1}{\sin\theta} &= \frac{[a^{\frac{2}{3}} + b^{\frac{2}{3}}]^{\frac{1}{2}}}{b^{\frac{1}{3}}} \\
\text{and } \frac{1}{\cos\theta} &= \frac{[a^{\frac{2}{3}} + b^{\frac{2}{3}}]^{\frac{1}{2}}}{a^{\frac{1}{3}}}.
\end{aligned}$$



Let L_{min} be the minimum value of L.

$$\begin{aligned}
L_{\text{min}} &= \frac{b}{\sin\theta} + \frac{a}{\cos\theta} \\
&= \frac{b[a^{\frac{2}{3}} + b^{\frac{2}{3}}]^{\frac{1}{2}}}{b^{\frac{1}{3}}} + \frac{a[a^{\frac{2}{3}} + b^{\frac{2}{3}}]^{\frac{1}{2}}}{a^{\frac{1}{3}}} \\
&= b^{\frac{2}{3}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}} + a^{\frac{2}{3}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
&= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{1}{2}} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}.
\end{aligned}$$

Hence the maximum length of pole that can be moved around the corner is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Although we have worked through the steps in creating the model sequentially, this may not happen when you are working on your own real-life problems. For example, you may need to add or modify an assumption if you come to a halt in the construction of the model, or you may need to add extra variables. This type of recursive procedure is not usually employed in tackling the more traditional mathematical problems in textbooks! Also you may decide to use a numerical method (for example using mathematical software) rather than trying to find an analytic solution by hand.

The next stage in the modelling cycle is to:

Interpret results

First, L should be a length measured in metres. Does our expression for the minimum value of L satisfy this criteria? As a and b have dimensions of length (in this case metres), $a^{\frac{2}{3}}$ and $b^{\frac{2}{3}}$ will have dimensions in $\mathrm{m}^{\frac{2}{3}}$. Thus L_{min} will have dimensions in $(\mathrm{m}^{\frac{2}{3}}))^{\frac{3}{2}}$ or m. So our result in dimensionally consistent.

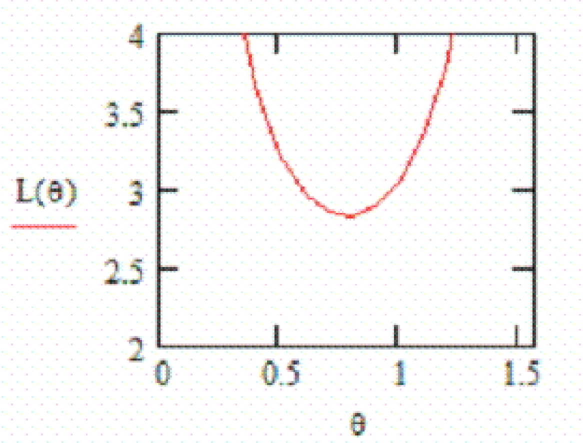
Does the solution seem reasonable given the physical constrains?

If a or b were very small, there would be very little space to manoeuvre and you would expect the maximum length of the pole to be close to the width of the larger corridor. This agrees with the result since if $b \to 0, L_{\text{min}} \to a$ and vice-versa.

You would also expect the maximum length of the pole to be greater than the width of each corridor, i.e. $L_{\text{min}} > a$ and $L_{\text{min}} > b$.

Since $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} > \left(a^{\frac{2}{3}}\right)^{\frac{3}{2}} = a$ and similarly for b, this also ties in with the physical interpretation.

If the two corridors have the same width, then $b = a$ and $L_{\text{min}} = \sqrt{2}^{\frac{3}{2}} a \approx 2.83a$.



This shows the graph of L against θ where we have taken $a = b = 1$.
As you would expect the minimum value of L occurs when $\theta = \pi/4$. This gives $L \approx 2.83$.

So if the corridors are the same width, we would expect to be able to move a horizontal pole of length approximately 2.8 times the width of the corridor.

The final stage of the modelling cycle is to:

Evaluate the model

To evaluate the model, we need to compare the models predictions with reality.

For example suppose the corridors both have a width of 1m, the model predicts that the maximum pole length will be about 2.8m. But does that work in practice?

You may find that holding the pole horizontally, the width of the curtain rail is significant and a 2.8m pole is slightly too long. In which case you may want to modify your model to include the width of the curtain rail and remove the assumption that the width is negligible.

You may also find that you can manoeuvre a slightly longer pole round the corner by lifting one end and working in three dimensions rather than two. Again, you could modify the model and obtain more accurate results. On the other hand if you just want a rough guide to the length of pole, the results from this model may suffice.

When you are satisfied that the model gives the results that you need, sum up the work you have done in a report for those concerned.

...AND FINALLY

You may be able to extend your model to cover other situations, for example corners that are not square or other items of furniture such as a bed which have a width.

HAPPY MODELLING!

End of main text. You can now close this page and return to the home page where you can choose to visit a different section.

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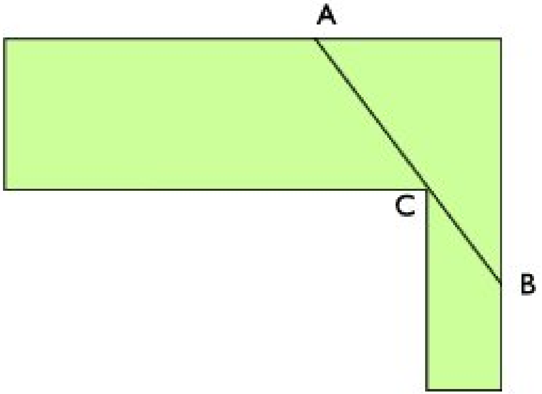
The restrictions are the two outside walls and the inner corner. In the limiting position, the pole would have its ends on the two outer walls (A and B) and would also touch the inner corner (C) as shown in diagram below:



If you are not convinced, sketch out the plan of the corridors on a piece of paper and using a thin rectangle of paper to represent the pole, try to slide the pole around the corner.

OK

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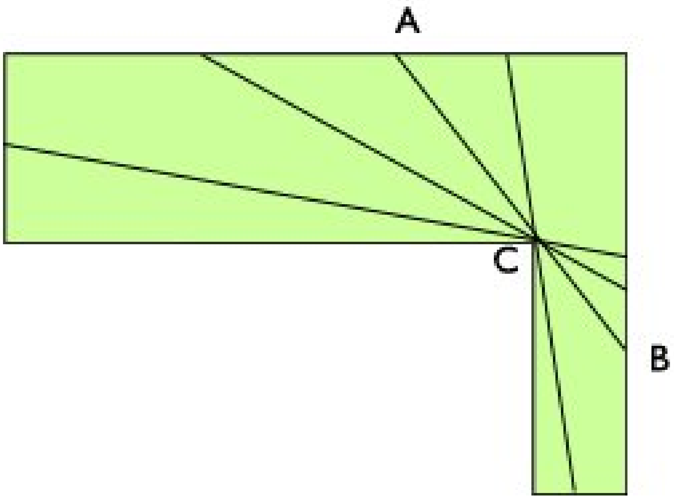
You should find that as the angle varies, the length of AB varies from large values (when AB is almost parallel to the walls) to a smaller minimum value. This MINIMUM value is the one we want to find as it will correspond to the MAXIMUM length of the pole that can be moved around the corner.



Click on this to download a java program illustrating the problem. If you cannot open this from here the URL is `http://maths-study-skills.open.ac.uk/p4/Pole.jar` OK

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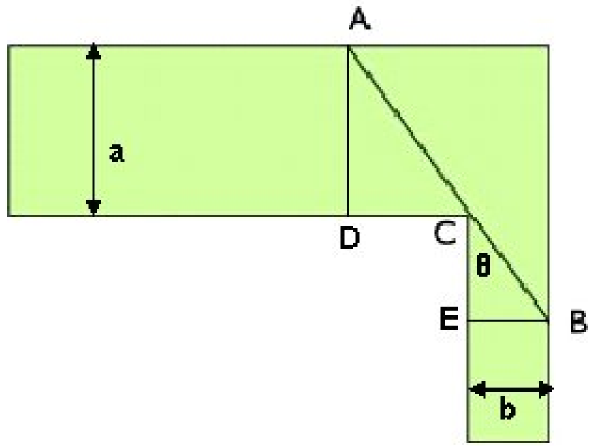
Here are some further assumptions we thought of. You may have some different ideas and this should be expected.

- Where two walls meet, these are at a perfect right angle?
- The pole does not bend or shorten.
- There are no obstacles such as rails, pictures, fire hydrants etc. in the vicinity of the bend.

OK

Thinks

As the adage says, 'One picture is worth a thousand words.' Draw a diagram to illustrate these variables.



We defined the following variables:

- a is the width of the first corridor (m)
- b is the width of the second corridor (m)
- θ is the angle AB makes with the inner wall of the second section. $(0 < \theta < \frac{\pi}{2})$
- L is the length of AB (m)

Notice that we do not consider data at this stage, as we want to construct a general mathematical model which can be used for corridors of different widths, in other buildings for instance.

OK

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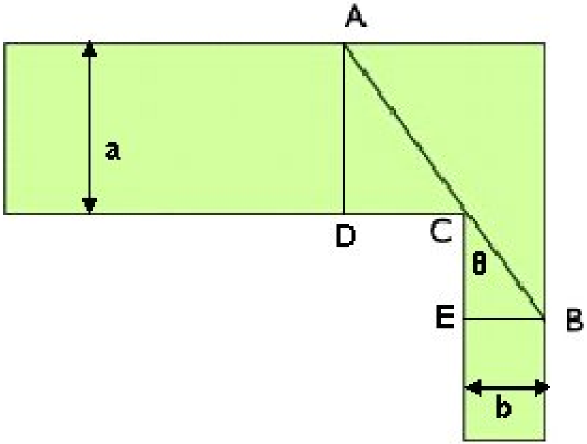
This is our approach:

$\angle DCA + \angle DCE + \angle ECB = \pi$ since these angles lie in a straight line.

$$DCA = \frac{\pi}{2} - \theta.$$

$$\sin\theta = \frac{EB}{BC} \text{ so } BC = \frac{b}{\sin\theta}.$$

$$\text{Similarly } CA = \frac{a}{\cos\theta}.$$



Thus $L = \dfrac{a}{\cos\theta} + \dfrac{b}{\sin\theta}$

or $L = a\sec\theta + b\,\mathrm{cosec}\,\theta$

since $\dfrac{1}{\cos\theta} = \sec\theta$ and $\dfrac{1}{\sin\theta} = \mathrm{cosec}\,\theta$.

OK

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Here is our solution.

$$\begin{aligned}
L &= \frac{a}{\cos\theta} + \frac{b}{\sin\theta} \\
\frac{dL}{d\theta} &= \frac{a\sin\theta}{\cos^2\theta} + \frac{-b\cos\theta}{\sin^2\theta}. \\
\text{So } \frac{dL}{d\theta} &= 0 \text{ when } \frac{b\cos\theta}{\sin^2\theta} = \frac{a\sin\theta}{\cos^2\theta},
\end{aligned}$$

that is when $b\cos^3\theta = a\sin^3\theta$ or $\tan^3\theta = \frac{b}{a}$.

So $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$. (In other notation: $\theta = \arctan\sqrt[3]{\left(\frac{b}{a}\right)}$.) Once the values of a and b are known, this equation can be solved. The solution we require will satisfy $0 < \theta < \frac{\pi}{2}$.

OK