Taylor polynomials of degree n about a

In an earlier section we considered polynomials about zero.

However it is possible to generalize the derivation of the formula for a Taylor polynomial about zero that you saw earlier to obtain a formula for a polynomial that approximates a suitable function f close to any chosen point a in its domain. The required formula is given below

Here suitable means that f must be differentiable at a the required number of times.

Taylor polynomials about a

Let f be a function that is n-times differentiable at a. The **Taylor polynomial** of degree n about a for f is

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$$

$$+ \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

When $a = 0$, this reduces to the formula for Taylor polynomials about 0. Note that the expression $f^{(n)}(a)$ denotes the value of the n'th derivative of f at a.

As in the case $a = 0$ the formula shows that any Taylor polynomial about a point a for a function f can be obtained from a Taylor polynomial of lower degree about a for f by adding the appropriate further terms. Also, again as in the case $a = 0$ it is possible for the Taylor polynomial of degree n about a for f to be of degree less than n; this happens when $f^{(n)} = 0$.

If p is a Taylor polynomial for a function f about a point a in its domain, and x is a point close to a, then $p(x)$ is an approximation for $f(x)$. Usually the greater the degree of the Taylor polynomial, and the closer x is to a, the more accurate the approximation.

Example Find the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$.

Solution The first four derivatives of the function $f(x) = \ln x$ are as follows:

$$f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}, f^{(3)}(x) = \frac{2}{x^3}, f^{(4)}(x) = -\frac{3 \times 2}{x^4} = -\frac{3!}{x^4}.$$

The product 3×2 has been simplified to 3! rather than 6 here to highlight the patterns emerging.

Evaluating f and its first four derivatives at 1 gives:

$$f(1) = 0, f'(1) = 1, f''(1) = -1, f^{(3)}(1) = 2, f^{(4)}(1) = -3!.$$

So by the formula above, the quartic Taylor polynomial about 1 for the function $f(x) = \ln x$ is

$$0 + 1 \times (x-1) + \frac{(-1)}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{(-3!)}{4!}(x-1)^4;$$

that is,

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4.$$

Notice that we do not multiply out the brackets in this expression. We usually leave a Taylor polynomial about a point a as a sum of terms each of which is the product of a constant and a power of $x - a$.

HAPPY READING!

End of main text. You can now close this page and return to the home page where you can choose to visit a different section.

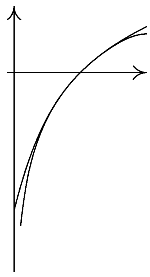
Note

This is a boxed formula. It is obviously important. We do have a handbook so I won't need to memorize this. It will help to have a feel for it. Is there a pattern? It is a Taylor polynomial about a. Each term, after the first, is a power of $(x - a)$ – indeed so is the first if I think of $(x - a)^0 = 1$. The coefficients are derivatives, evaluated at a, divided by some factorial. The last term gives the pattern – the nth derivative divided by $n!$. So you need to be able to differentiate the function n times to find this formula. Ah! The preamble says 'Let f be a function that is differentiable n times (at a)'. OK. I'll want to calculate this quite soon. I don't fancy differentiating a rational function n times! There 's probably an example using this below but can I try one first. Something easy, like $f(x) = x$. Not about 0 that would be silly. Ok. what about $a = 1$. After the first derivative everything is zero (that's easy). Now $f(1) = 1$, $f'(x) = 1$ so $f'(1) = 1$. The polynomial is $p(x) = 1 + 1(x - 1)$. How silly but it does work!

OK

Scribbles

I did that. This seems straightforward. I wonder what the graphs look like.



Not bad! Such a graph would probably need to be done using an aid such as MathCad or graphical calculator.

$\boxed{\text{OK}}$